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Periodic Solutions of Some Stationary Linear Systems

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1. INTRODUCTION

In this paper we discuss the existence of periodic solutions of some stationary linear nonhomogeneous systems. For the ordinary differential systems, as well as for the difference-differential ones, the results exposed here are well known and can also be obtained by other means. For instance, a method (used for variable coefficients too) consists in the examination of translation by a period along the solutions of the system, this translation operator being a compact one (see, for example, [*3*]). However for other systems, and we are interested first of all in systems of the neutral type, this method is of no use because the translation operator is, generally, not compact. That is why we shall consider another operator U defined on the space of 2π-periodic C^∞ functions. An operator of the same kind is used for many purposes in the theory of partial differential equations. The adjoint operator ${}^tU$ is defined over a space of periodic distributions and we shall be interested in the solutions of the equation ${}^tUy = f$. For this purpose we establish some conditions which imply that U should be a homomorphism[1]. The utility of homomorphisms when studying equations derives from the following extension of Banach's theorem (see [*2*]): *Let E, F be Frechet spaces and $U : E \to F$ a linear operator. The following assertions are equivalent*:

(α) *U is a homomorphism*;

(β) *$U(E)$ is closed in F*;

(γ) *U is continuous and possesses a continuous right inverse* i.e., *there is a continuous linear operator $V : U(E) \to E$ so that $UVf = f$ for all $f \in U(E)$*;

(δ) *U is continuous and for $f \in E'$ the equation ${}^tUy = f$ has a solution if and only if f is orthogonal to every solution of the homogeneous equation $Ux = 0$*[2].

[1] According to [*2*] a linear operatur U from the locally convex space E into the locally convex space F is called a *homomorphism*, if the canonical algebraic isomorphism $E/U^{-1}(0) \to F$ is also a topological one. This holds if and only if U is continuous and $U(G)$ is open in $U(E)$ for any open set G.

[2] E', F' are the adjoint spaces of E, F.

The use of distributions is essential in our method, for the adjoint of the function space quoted above is a distribution space. Thus the solutions of the equation ${}^{t}Uy = f$ are generalized solutions. For systems of neutral type the framework of distributions seems to be useful in any case, because the simplest neutral systems have generalized solutions. The scalar equation

$$\dot{x}(t) - \dot{x}(t-h) = 0$$

is satisfied by any h-periodic distribution. We discuss also the existence of solutions of class C^∞.

The homomorphism condition for stationary operators which we shall find is given in terms of the Fourier series (transform) of a certain distribution related to the operator U; i.e. it is given by the asymptotical behavior of the characteristic function on the integer points of the imaginary axis. For systems of neutral type this condition is determined by the behavior of some quasipolynomials on the imaginary axis (see Section 4) and sometimes it is connected to problems of number theory.

In Section 2 we recall some properties concerning the space of periodic distributions introduced in [7] and we give some simple extensions. In Section 3 we establish necessary and sufficient conditions for an operator to be a homorphism. In Section 4 we discuss systems of the neutral type.

The method exposed here can be extended without difficulties for studying the existence of periodic solutions of partial differential equations which are periodic with respect to all variables. It seems that this method is also useful for discussing the existence of such solutions which, in addition, are odd with respect to some variables. This kind of question may arise if discussing the existence of periodic solutions of some boundary-value problems (see [*4*, *8*]).

2. Basic Spaces

We designate by $\mathscr{R}^l$ respectively $\mathscr{C}^l$ the real, respectively complex, l-dimensional space normed by the Euclidian norm. Let $\mathscr{P}$ be the space of functions $\mathscr{R}^1 \to \mathscr{C}^1$ of class C^∞, which are 2π-periodic. We define the locally convex topology of $\mathscr{P}$ by the norms $\|\cdot\|_m$, $m = 0, 1, 2,\ldots$, where

$$\|\varphi\|_m = \max_{0\leqslant j\leqslant m} \sup_{t\in\mathscr{R}^1} |D^j\varphi(t)|, \qquad \text{for} \quad \varphi\in\mathscr{P},$$

D^j being the differential operator of order j. $\mathscr{P}$ is a Fréchet space and also a Montel space. In the adjoint space $\mathscr{P}'$ we introduce the strong topology. $\mathscr{P}'$ is isomorphic to the 2π-periodic distributions subspace of the space $\mathscr{D}'$ of all the distributions (see [7]). We shall call the elements of $\mathscr{P}'$, 2π-periodic distributions. The multiplication by functions, the translation and the derivation are defined in $\mathscr{P}'$ to be the usual methods of the theory of distributions.

Let g be an element of $\mathscr{P}'$. For any $\varphi \in \mathscr{P}$ we put

$$\psi(s) = \langle \varphi(t + s), g(t) \rangle.$$

It is easy to verify that ψ is an element of $\mathscr{P}$ and that the linear operator $\varphi \to \psi$ of $\mathscr{P}$ into $\mathscr{P}$ is a continuous one. The adjoint of this operator $\mathscr{P}' \to \mathscr{P}'$ is the convolution operator by g. Its value on a distribution $f \in \mathscr{P}'$ is denoted by $g * f$ and is called the *convolution of the distribution f by g*. Thus the convolution $g * f$ is defined for $\varphi \in \mathscr{P}$ by

$$\langle \varphi, g * f \rangle = \langle \langle \varphi(t + s), g(t) \rangle, f(s) \rangle.$$

This convolution is defined for any pair of distributions f, g belonging to $\mathscr{P}'$. The operator $\varphi \to \psi$ may also be represented as a convolution $\psi = \check{g} * \varphi$, where $\check{}$ is the symmetry: $\check{\varphi}(t) = \varphi(t)$, if $\varphi \in \mathscr{P}$ and if $f \in \mathscr{P}'$, then $\check{f} \in \mathscr{P}'$ is defined by

$$\langle \varphi, \check{f} \rangle = \langle \check{\varphi}, f \rangle, \qquad \text{for} \qquad \varphi \in \mathscr{P}.$$

The convolution defined above is associative and commutative. Also if τ_h is the translation by h,

$$(\tau_h \varphi)(t) = \varphi(t - h), \qquad \text{for} \qquad \varphi \in \mathscr{P},$$

$$\langle \tau_h f, \varphi \rangle = \langle f, \tau_{-h} \varphi \rangle, \qquad \text{for} \qquad \varphi \in \mathscr{P}, \quad f \in \mathscr{P}'$$

and δ is the measure of Dirac, then $D(g * f) = (Dg) * f$, $\tau_h(g * f) = (\tau_h g) * f$, $\delta * f = f$.

The Fourier series of $\varphi \in \mathscr{P}$,

$$\sum_n c_n e^{int}, \qquad c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi(t)\, e^{-int}\, dt, \quad n = 0, \pm 1, \pm 2, \ldots,$$

converge in $\mathscr{P}$ to φ and the system $\{e^{int}\}$ is a basis in $\mathscr{P}$. We have immediately that *if $\varphi \in \mathscr{P}$*, then

$$\lim_{|n| \to \infty} |n|^p \, |c_n| = 0, \qquad \text{for all} \quad p \geqslant 0. \tag{1}$$

Conversely, if the two-sided numerical sequence $\{c_n\}$ verifies (1), *then the series $\sum_n c_n e^{int}$ converge to a function $\varphi \in \mathscr{P}$.*

The Fourier coefficients of a distribution $f \in \mathscr{P}'$ are

$$c_n = \frac{1}{2\pi} \langle e^{-int}, f(t) \rangle$$

and the Fourier series $\sum_n c_n e^{int}$ converge in $\mathscr{P}'$ to f (see [7]). The system $\{e^{int}\}$ is a basis in $\mathscr{P}'$. *For any $f \in \mathscr{P}'$, there is $p \geqslant 0$, so that the Fourier coefficients of f verify*

$$\lim_{|n|\to\infty} |n|^{-p} |c_n| = 0. \tag{2}$$

Conversely, let $\{c_n\}$ be a two-sided numerical sequence; if there is $p \geqslant 0$, so that (2) *holds, then $\sum_n c_n e^{int}$ converge to a distribution of $\mathscr{P}'$* (see [7]).

The Fourier coefficients of a convolution are given by

$$c_n(f * g) = 2\pi c_n(f)\, c_n(g).$$

The following proposition is an analog of a result established in [7] for $\mathscr{D}'$.

PROPOSITION 1. *If U is a continuous linear operator $\mathscr{P} \to \mathscr{P}$, the following assertions are equivalent*:

(i) *U is stationary* (i.e. *$\tau_h U = U\tau_h$ for any h*);

(ii) *U is a convolution operator*; i.e., *there is $r \in \mathscr{P}'$ so that*

$$U\varphi = \check{r} * \varphi, \qquad \textit{for any} \qquad \varphi \in \mathscr{P};$$

(iii) *U commutes with the differential operator, $DU = UD$*;

(iv) *e^{int} are eigenfunctions of U*; i.e.,

$$Ue^{int} = a_n e^{int} \tag{4}$$

Moreover if (iv) *holds, then $(2\pi)^{-1} a_n$ are the Fourier coefficients of the distribution $\check{r}$.*

Proof. (i) $\Rightarrow$ (ii). We first show that

$${}^tU(g * f) = ({}^tUg) * f, \qquad \text{for any} \qquad g, f \in \mathscr{P}'. \tag{5}$$

In fact, for $\varphi \in \mathscr{P}$, we have

$$\begin{aligned}\langle \varphi, {}^tU(g * f)\rangle &= \langle U\varphi, g * f\rangle = \langle\langle (U\varphi)(t+s), g(t)\rangle, f(s)\rangle \\ &= \langle\langle (\tau_{-s}U\varphi)(t), g(t)\rangle, f(s)\rangle = \langle\langle (U\tau_{-s}\varphi)(t), g(t)\rangle, f(s)\rangle \\ &= \langle\langle (\tau_{-s}\varphi)(t), ({}^tUg)(t)\rangle, f(s)\rangle = \langle\langle \varphi(t+s), ({}^tUg)(t)\rangle, f(s)\rangle \\ &= \langle \varphi, ({}^tUg) * f\rangle\end{aligned}$$

and (5) holds. For $f \in \mathscr{P}'$, this yields ${}^tUf = {}^tU(\delta * f) = ({}^tU\delta) * f$ and thus ${}^tUf = r * f$, for any $f \in \mathscr{P}'$, where we have put $r = {}^tU\delta$. Thus, $U\varphi = \check{r} * \varphi$, for $\varphi \in \mathscr{P}$.

(ii) $\Rightarrow$ (iii). In fact, for $\varphi \in \mathscr{P}$, we have

$$DU\varphi = D(\check{r} * \varphi) = \check{r} * (D\varphi) = UD\varphi.$$

(iii) $\Rightarrow$ (iv). We put $\psi = Ue^{int}$. Then,

$$D\psi = DUe^{int} = UDe^{int} = Uine^{int} = inUe^{int} = in\psi.$$

The solutions of the differential equation $D\psi = in\psi$ are $\psi(t) = ae^{int}$ and (iv) holds.

(iv) $\Rightarrow$ (i). We have

$$\begin{aligned} U\tau_h e^{int} &= Ue^{in(t-h)} = Ue^{-inh}e^{int} = e^{-inh}Ue^{int} \\ &= e^{-inh}a_n e^{int} = \tau_h(a_n e^{int}) = \tau_h(Ue^{int}). \end{aligned}$$

Thus U and τ_h commute on the elements of the basis $\{e^{int}\}$ and this yields $\tau_h U^\varphi = U\tau_h$ for all.

We now assume that (iv) holds. By (3) we obtain

$$Ue^{int} = \check{r} * e^{int} = \sum_n c_n(\check{r} * e^{int})\, e^{int} = \varphi \in \mathscr{P}.$$

$$\sum_n c_n(\check{r})\, c_n(e^{int})\, e^{int} = 2\pi \sum_n c_n(\check{r})\, e^{int}.$$

By comparing with (4) we deduce that $2\pi c_n(\check{r}) = a_n$ and this achieves the proof.

As we have seen in the above proof, the distribution r of the condition (ii) is uniquely determined by $r = {}^tU\delta$. We note also that analogous properties may be established for the operators $\mathscr{P}' \to \mathscr{P}'$.

We also consider vector distributions with values in $\mathscr{C}^l$. Let $\mathscr{P}^l$ be the topological product of l spaces equal to $\mathscr{P}$. We designate the elements of $\mathscr{P}^l$ by the column vectors, $\varphi = {}^t(\varphi_1, ..., \varphi_l)$, $\varphi_j \in \mathscr{P}$. The elements of the adjoint space $\mathscr{P}'^l$ are the column vectors $f = {}^t(f_1, ..., f_l)$, $f_j \in \mathscr{P}'$, the scalar product being defined by

$$\langle \varphi, f \rangle = \sum_{j=1}^{l} \langle \varphi_j, f_j \rangle.$$

A linear operator $A : \mathscr{P}^l \to \mathscr{P}^l$ is represented, by an $l \times l$-matrix $A = (A_{ij})$, where A_{ij} are linear operators $\mathscr{P} \to \mathscr{P}$ and for $\varphi \in \mathscr{P}^l$, $\varphi = {}^t(\varphi_1, ..., \varphi_l)$, we have

$$A\varphi = {}^t\left(\sum_{j=1}^{l} A_{ij}\varphi_j, ..., \sum_{j=1}^{l} A_{lj}\varphi_j\right).$$

Obviously, A is continuous if and only if the A_{ij} are continuous. The same remarks are also true for the linear operators $\mathscr{P}'^l \to \mathscr{P}'^l$. We have immediately that if $A = (A_{ij})$ is a linear continuous operator $\mathscr{P}^l \to \mathscr{P}^l$, then the adjoint operator ${}^tA : \mathscr{P}'^l \to \mathscr{P}'^l$ has the matrix ${}^tA = ({}^tA_{ji})$. If $F = (f_{ij})$ is a distribution matrix, $f_{ij} \in \mathscr{P}'$, we define the convolution $F * g$, for $g \in \mathscr{P}'^l$, $g = {}^t(g_1, \ldots, g_l)$, $g_j \in \mathscr{P}'$, by

$$F * g = {}^t\left(\sum_{j=1}^{l} f_{1j} * g_j, \ldots, \sum_{j=1}^{l} f_{lj} * g_j\right).$$

The operator $g \to F * g$ is the adjoint of $\varphi \to \check{F} * \varphi$ defined on $\mathscr{P}^l$, where $\check{F} = (\check{f}_{ji})$.

The subspace $\mathscr{V}^n$ of all the vectors ce^{int}, $c = {}^t(c_1, \ldots, c_l)$ is isomorphic to the complex Euclidean space $\mathscr{C}^l$. We consider in $\mathscr{V}^n$ the Euclidean norm

$$| ce^{int} | = \left(\sum_{j=1}^{l} | c_j |^2\right)^{1/2}.$$

From the Fourier expansion we deduce that any element of $\mathscr{P}^l$ (respectively $\mathscr{P}'^l$) has a unique representation as a series with terms belonging to $\mathscr{V}^n$. The assertions concerning the behavior of the Fourier coefficients of scalar functions and distributions [formulae (1) and (2)] also remain valid for vector functions and distributions; one must only replace the modulus by the norms of the vector Fourier coefficients.

Let $F = (f_{ij})$ be a distribution matrix, $f_{ij} \in \mathscr{P}'$. We say that the matrix $C_n = (c_n^{ij})$, $c_n^{ij} = c_n(f_{ij})$ are the Fourier coefficients of F. There is $p \geqslant 0$, so that[3]

$$\lim_{|n| \to \infty} | n |^{-p} | C_n | = 0. \tag{6}$$

Let $\{C_n\}$ be a sequence of numerical matrices; if there exists $p \geqslant 0$ so that (6) *holds, then there is a unique distribution matrix $F = (f_{ij})$ with C_n as Fourier coefficients.*

One immediately verifies that for $F = (f_{ij})$, $f_{ij} \in \mathscr{P}'$, $g \in \mathscr{P}'^l$ we have

$$C_n(F * g) = 2\pi \, C_n(F) \, c_n(g) \tag{7}$$

and thus

$$| C_n(F * g)| \leqslant 2\pi \, | C_n(F)| \, | c_n(g)|. \tag{8}$$

Proposition 1 also remains valid for the operators $U : \mathscr{P}^l \to \mathscr{P}^l$, the condition (iv) being replaced by

(iv′) *the subspaces $\mathscr{V}^n$, $n = 0, \pm 1, \pm 2, \ldots$, are invariant for U*; i.e., *there are numerical matrices A_n so that*

$$Uce^{int} = A_n ce^{int}.$$

Moreover, if (iv′) *holds, then $(2\pi)^{-1} A_n$ are the Fourier coefficients of $\hat{R} = (\check{r}_{ji})$.*

[3] The norm of a matrix C is $| C | = \Lambda$, where Λ^2 is the greatest eigenvalue of C^*C.

3. Stationary Homomorphisms

Let $U : \mathscr{P}^l \to \mathscr{P}^l$ be a continuous stationary linear operator. By Proposition 1 extended to operators $\mathscr{P}^l \to \mathscr{P}^l$, there is a unique matrix of distributions $R = (r_{ij})$, $r_{ij} \in \mathscr{P}'$ so that $U\varphi = \check{R} * \varphi$, for $\varphi \in \mathscr{P}^l$ and ${}^tUf = R * f$, for $f \in \mathscr{P}'^l$. We denote the Fourier coefficients of $\check{R}$ by $C_n = (c_n^{ij})$, where c_n^{ij} are the Fourier coefficients of $\check{r}_{ij}$,

$$c_n^{ij} = c_n(\check{r}_{ij}) = \frac{1}{2\pi} \langle e^{-int}, \check{r}_{ij}(t) \rangle.$$

The Fourier coefficients of r_{ij} satisfy $c_n(r_{ij}) = c_{-n}(\check{r}_{ij})$, and we deduce that the Fourier coefficients of R are the matrices $(c_{-n}^{ji}) = {}^tC_{-n}$.

By (7) follows that if

$$\varphi = \sum_n c_n(\varphi)\, e^{int}, \varphi \in \mathscr{P}^l,$$

then

$$U\varphi = \sum_n C_n c_n(\varphi)\, e^{int}, \tag{9}$$

and if

$$f = \sum_n c_n(f)\, e^{int}, f \in \mathscr{P}'^l,$$

then

$${}^tUf = \sum_n {}^tC_{-n} c_n(f)\, e^{int}. \tag{10}$$

Since (9) and (10) hold, the kernel of U (respectively ${}^tU$) is equal to the closed-in $\mathscr{P}^l$ (resp. $\mathscr{P}'^l$) linear envelope of all kernels of U (resp. ${}^tU$) relative to $\mathscr{V}^n$. The kernel of U (resp. ${}^tU$) relative to $\mathscr{V}^n$ is the subspace of the vectors ce^{int} verifying $C_n c_n = 0$ (resp. ${}^tC_{-n}c_n = 0$). The matrices C_n and ${}^tC_n$ have the same rank and we have

Proposition 2. *The kernels of U (in $\mathscr{P}^l$) and ${}^tU$ (in $\mathscr{P}'^l$) are of the same dimension.*

If one of these kernels is not of finite dimension, Proposition 2 means that both are not of finite dimension.

Theorem 1. *Let U be a stationary homomorphism. Then ${}^tU\mathscr{P}'^l = \mathscr{P}'^l$ if and only if the kernel of ${}^tU$ (in $\mathscr{P}'^l$) is zero. If this condition holds, then ${}^tU$ is an algebraic and topological isomorphism of $\mathscr{P}'^l$ onto $\mathscr{P}'^l$.*

Proof. Suppose that ${}^tU\mathscr{P}'^l = \mathscr{P}'^l$. $U^{-1}(0) = \{0\}$ for $U^{-1}(0)$ is the orthogonal complement of ${}^tU\mathscr{P}'^l$. By Proposition 2, the kernel of ${}^tU$ is also zero.

Reciprocally, if the kernel of ${}^tU$ is zero, by Proposition 2 we have $U^{-1}(0) = \{0\}$. From Banach's theorem (Section 1) we have ${}^tU\mathscr{P}'^l = \mathscr{P}'^l$.

Suppose now that the condition of the theorem holds. We shall see that $U\mathscr{P}^l = \mathscr{P}^l$. In fact, the kernel of ${}^tU$ (being equal to zero), the orthogonal complement of $U\mathscr{P}^l$, and a corollary of the Hahn–Banach theorem imply that $U\mathscr{P}^l$ is dense in $\mathscr{P}^l$. The condition (β) of Banach's theorem implies $U\mathscr{P}^l = \mathscr{P}^l$. From $U^{-1}(0) = \{0\}$ we deduce that U has an inverse $U^{-1} : \mathscr{P}^l \to \mathscr{P}^l$. From Banach's theorem it follows that U^{-1} is continuous. Then ${}^t(U^{-1}) = ({}^tU)^{-1}$ and this achieves the proof.

Condition (γ) of Banach's theorem and the theorem exposed above show the utility of homomorphisms when studying equations of the type ${}^tUy = f$. We now discuss conditions for an operator to be a homomorphism. We first formulate a lemma of linear algebra.

Let $A : \mathscr{C}^l \to \mathscr{C}^l$ be a linear operator $\neq 0$, A^* the adjoint operator, $\mathscr{R}(A)$ the range of A and $\mathscr{N}(A)$ the kernel of A. The space $\mathscr{C}^l$ can be represented as an orthogonal sum

$$\mathscr{C}^l = \mathscr{R}(A) \oplus \mathscr{N}(A^*) \qquad \text{and} \qquad \mathscr{C}^l = \mathscr{R}(A^*) \oplus \mathscr{N}(A). \tag{11}$$

It is immediate that

$$A\mathscr{R}(A^*) = \mathscr{R}(A).$$

$\mathscr{R}(A^*)$ and $\mathscr{R}(A)$ are subspaces of the same dimension; thus the restriction B of A to $\mathscr{R}(A^*)$ is an invertible operator. Let λ^2 be the smallest eigenvalue $\neq 0$ of A^*A.

LEMMA 1. *The operator $B : \mathscr{R}(A^*) \to \mathscr{R}(A)$ is invertible and its inverse $B^{-1} : \mathscr{R}(A) \to \mathscr{R}(A^*)$ has the norm $| B^{-1} | = 1/\lambda$.*

Proof. This is obvious since $\sigma(B^*B) = \sigma(A^*A) - \{0\}$ and $\sigma(B^{*-1}B^{-1}) = \{1/\lambda, \lambda \in \sigma(B^*B)\}$ and $\| C \| = \sup\{\lambda; \lambda \in \sigma(C^*C)\}$. σ denotes the spectrum.

Let $U : \mathscr{P}^l \to \mathscr{P}^l$ be a continuous stationary linear operator given by the matrix of distributions $\check{R}$ and let C_n be the Fourier coefficients of $\check{R}$. We designate by λ_n^2 the smallest eigenvalue $\neq 0$ of C^*C_n (if $C_n = 0$, we put $\lambda_n = 0$).

PROPOSITION 3. *The continuous stationary linear operator $U : \mathscr{P}^l \to \mathscr{P}^l$ is a homomorphism if and only if there is $K > 0$ and $p \geqslant 0$, so that*

$$\lambda_n \geqslant K \mid n \mid^{-p} \qquad \textit{for any} \qquad n, \lambda_n \neq 0. \tag{12}$$

Proof. The subspace $\mathscr{V}^n$ is the orthogonal sum

$$\mathscr{V}^n = \mathscr{R}(C_n^*) \oplus \mathscr{N}(C_n) \qquad \text{and} \qquad \mathscr{V}^n = \mathscr{R}(C_n) \oplus \mathscr{N}(C_n^*). \tag{13}$$

Let B_n be the restriction of C_n to $\mathscr{R}(C_n^*)$. By Lemma 1, B_n is an isomorphism of $\mathscr{R}(C_n^*)$ onto $\mathscr{R}(C_n)$ and $| B_n^{-1} | = 1/\lambda_n$, if $B_n \neq 0$.

Let $\mathscr{R}_1, \mathscr{N}_1, \mathscr{R}_2$ respectively $\mathscr{N}_2$, be closed in $\mathscr{P}^l$ linear envelopes of $\mathscr{R}(C_n^*), \mathscr{N}(C_n)$, $\mathscr{R}(C_n)$ respectively $\mathscr{N}(C_n^*)$. Obviously $\mathscr{N}_1 = U^{-1}(0)$.

The space $\mathscr{P}^l$ can be represented as a topological direct sum:

$$\mathscr{P}^l = \mathscr{R}_1 \dotplus \mathscr{N}_1 \quad \text{and} \quad \mathscr{P}^l = \mathscr{R}_2 \dotplus \mathscr{N}_2 . \tag{14}$$

In fact, let φ be any element of $\mathscr{P}^l$, $\varphi = \sum_n c_n e^{int}$. By $c_n \in \mathscr{V}^n$ and by (13) follows the unique decomposition $c_n = c_n' + c_n''$, where $c_n' \in \mathscr{R}(C_n^*)$, $c_n'' \in \mathscr{N}(C_n)$. Since

$$| c_n |^2 = | c_n' |^2 + | c_n'' |^2,$$

the series

$$\sum_n c_n' e^{int}, \quad \sum_n c_n'' e^{int}$$

are convergent in $\mathscr{P}^l$. Thus for any $\varphi \in \mathscr{P}^l$ we have

$$\varphi = \varphi_1 + \varphi_2 , \quad \varphi_1 \in \mathscr{R}_1 , \varphi_2 \in \mathscr{N}_2 \quad \text{where} \quad \varphi_1 = \sum_n c_n' e^{int}, \quad \varphi_2 = \sum_n c_n'' e^{int}$$

and, obviously, this decomposition is unique. This yields that $\mathscr{P}^l$ is an algebraic direct sum of the closed subspaces $\mathscr{R}_1$ and $\mathscr{N}_1$. Then the first of formulae (14) derives from [2] since $\mathscr{P}^l$ is a Fréchet space. The second of the formulae (14) can be obtained by analogous means.

Let us observe that

$$\overline{U\mathscr{P}^l} = \mathscr{R}_2 . \tag{15}$$

Assume that condition (12) holds. We shall prove that $U\mathscr{P}^l = \mathscr{R}_2$. If ψ is any element of $\mathscr{R}_2$, then $\psi = \sum_n c_n e^{int}$, and $c_n \in \mathscr{R}(C_n)$. If $b_n = B_n^{-1} c_n$, then $b_n \in \mathscr{R}(C_n^*)$. If $b_n \neq 0$, then

$$| b_n | = | B_n^{-1} c_n | \leqslant | B_n^{-1} || c_n | \leqslant \frac{1}{\lambda_n} | c_n | \leqslant \frac{1}{K} | n |^p | c_n | .$$

Thus, for any $q \geqslant 0$, $\lim_{n\to\infty} | n |^q | b_n | = 0$. This implies the series $\sum_n b_n e^{int}$ is convergent in $\mathscr{P}^l$ to a function φ belonging to $\mathscr{R}_1$. Then

$$U\varphi = \sum_n C_n b_n e^{int} = \sum_n B_n b_n e^{int} = \sum_n c_n e^{int} = \psi$$

and $U\mathscr{P}^l = \mathscr{R}_2$. By (15), $U\mathscr{P}^l$ is closed and from the condition (β) of Banach's theorem we deduce that U is a homomorphism.

Assume now that U is a homomorphism. We show that (12) holds. By Banach's theorem, $U\mathscr{P}^l$ is closed and by (15), $U\mathscr{P}^l = \mathscr{R}_2$. Let B be the restriction of U to $\mathscr{R}_1$. By (14), $B\mathscr{R}_1 = U\mathscr{P}^l$ and thus $B\mathscr{R}_1 = \mathscr{R}_2$. Obviously,

B is invertible. By Banach's theorem the inverse $B^{-1} : \mathscr{R}_2 \to \mathscr{R}_1$ is continuous. Let $V : \mathscr{P}^l \to \mathscr{P}^l$ be an extension of B^{-1}, $V\varphi = B^{-1}\varphi$, for $\varphi \in \mathscr{R}_2$, $V\varphi = 0$, for $\varphi \in \mathscr{N}_2$. By (14), V is continuous. Moreover,

$$V\varphi = B_n^{-1}\varphi, \qquad \text{for} \qquad \varphi \in \mathscr{R}(C_n), \qquad V\varphi = 0, \qquad \text{for} \qquad \varphi \in \mathscr{N}(C_n^*).$$

By (13), the subspaces $\mathscr{V}^n$ are invariant relatively to V. If G_n is the restriction of V to $\mathscr{V}^n$, then $G_n : \mathscr{V}^n \to \mathscr{V}^n$ and

$$G_n\varphi = B_n^{-1}\varphi, \qquad \text{for} \qquad \varphi \in \mathscr{R}(C_n), \qquad G_n\varphi = 0, \qquad \text{for} \qquad \varphi \in \mathscr{N}(C_n^*).$$

Since $| G_n | = | B_n^{-1} |$, we have

$$| G_n | = \frac{1}{\lambda_n}, \qquad \text{if} \qquad \lambda_n \neq 0, \qquad | G_n | = 0, \qquad \text{if} \qquad \lambda_n = 0.$$

By Proposition 1 (extended to operators $\mathscr{P}^l \to \mathscr{P}^l$), V is a convolution operator with G_n as Fourier coefficients. By (6) we deduce that there is $p \geqslant 0$ so that $\lim_{|n| \to \infty} | n |^{-p} | G_n | = 0$. Thus, there is $K_1 > 0$, so that $| G_n | \leqslant K_1 | n |^p$, for $n \neq 0$. This means

$$\frac{1}{\lambda_n} \leqslant K_1 | n |^p$$

and the λ_n verify (12).

Remarks. (a) Let U be a stationary homomorphism and $f \in {}^tU\mathscr{P}'^l$. If the kernel of U is not zero, then the equation ${}^tUy = f$ does not have a unique solution. However, one can define a continuous linear operator $f \to y$. This is the restriction of ${}^tV$ (V given in the proof of the above theorem) to the subspace ${}^tU\mathscr{P}'^l$; i.e.,

$$ {}^tU{}^tVf = f, \qquad \text{for any} \qquad f \in {}^tU\mathscr{P}'^l. \tag{16}$$

In fact, by (14) and since $\mathscr{N}_1 = U^{-1}(0)$, $\mathscr{P}^l = \mathscr{R}_1 + U^{-1}(0)$. Thus any $\varphi \in \mathscr{P}^l$ can be represented as $\varphi = \varphi_1 + \varphi_2$, $\varphi_1 \in \mathscr{R}_1$, $\varphi \in U^{-1}(0)$. Since the kernel of U is the orthogonal complement of ${}^tU\mathscr{P}'^l$, we have

$$\langle \varphi_2, f \rangle = 0 \qquad \text{for} \qquad \varphi_2 \in U^{-1}(0), \quad f \in {}^tU\mathscr{P}'^l.$$

Then, for any $\varphi \in \mathscr{P}^l$ and $f \in {}^tU\mathscr{P}'^l$, we have

$$\begin{aligned} \langle \varphi, {}^tU{}^tVf \rangle &= \langle VU\varphi, f \rangle = \langle VU\varphi_1, f \rangle \\ &= \langle B^{-1}B\varphi_1, f \rangle = \langle \varphi_1, f \rangle = \langle \varphi_1 + \varphi_2, f \rangle = \langle \varphi, f \rangle. \end{aligned}$$

and (16) holds.

(b) The previous remark shows that the continuous linear operator ${}^tV : {}^tU\mathscr{P}'^l \to \mathscr{P}'^l$ is the right inverse of $U : \mathscr{P}'^l \to {}^tU\mathscr{P}'^l$. By [2], ${}^tU$ is also a homomorphism.

(c) It is easy to verify that the condition of Proposition 3 always holds for ordinary stationary differential equations, for difference-differential equations, as well as for integro-differential ones (see Section 4). The matrices C_n are the values of the characteristic function of the system on the integer points situated on the imaginary axis.

We further show that the homomorphism condition is equivalent to the existence of some regular solutions. First we establish a lemma.

Let $\{\nu_n\}_{n\geqslant 1}$ be a sequence of positive numbers with

$$\inf\{\nu_n , \nu_n \neq 0, n \geqslant 1\} = 0.$$

Let $\{\nu_{n_k}\}_{k\geqslant 1}$ be the subsequence defined as follows: ν_{n_1} is the first nonzero term of $\{\nu_n\}_{n\geqslant 1}$; ν_{n_2} is the first nonzero term smaller than ν_{n_1} and following ν_{n_1}; ν_{n_3} is the first nonzero term smaller than ν_{n_2} and following ν_{n_2}, and so forth. Obviously $\{\nu_{n_k}\}_{k\geqslant 1}$ is a strictly decreasing subsequence of $\{\nu_n\}_{n\geqslant 1}$ with $\nu_{n_k} \to 0$ as $k \to \infty$. We call $\{\nu_{n_k}\}_{k\geqslant 1}$ the *canonical subsequence of* $\{\nu_n\}_{n\geqslant 1}$.

Let $\{\nu_n\}_{n\geqslant 1}$ be a sequence of positive numbers and α be an integer $\geqslant 0$ with

$$\inf\{n^{\alpha+1}\nu_n , \nu_n \neq 0, n \geqslant 1\} = 0.$$

Obviously

$$\inf\{n^{\alpha}\nu_n , \nu_n \neq 0, n \geqslant 1\} = 0.$$

LEMMA 2. *If $\{n_k^{\alpha+1}\nu_{n_k}\}_{k\geqslant 1}$ is the canonical subsequence of $\{n^{\alpha+1}\nu_n\}_{n\geqslant 1}$ and $\{m_k^{\alpha}\nu_{m_k}\}_{k\geqslant 1}$ is the canonical subsequence of $\{n^{\alpha}\nu_n\}_{n\geqslant 1}$, then the sequence of indices $\{n_k\}_{k\geqslant 1}$ is a subsequence of $\{m_k\}_{k\geqslant 1}$.*

Proof. We show by induction that $n_j \in \{m_k\}_{k\geqslant 1}$ for any $j \geqslant 1$. In fact, $n_1 \in \{m_k\}_{k\geqslant 1}$ for $n_1 = m_1$. We show that if $n_i \in \{m_k\}_{k\geqslant 1}$, for $1 \leqslant i \leqslant j-1$ then n_j has the same property. If this doesn't hold then n_j would be situated between two consecutive numbers m_k : $m_h < n_j < m_{h+1}$. But our assumption implies the relation $m_h < n_{j-1} < m_{h+1}$ is not possible. Thus $n_{j-1} \leqslant m_h$. By the definition of $\{n_k^{\alpha+1}\nu_{n_k}\}_{k\geqslant 1}$ we deduce

$$n_{j-1}^{\alpha+1}\nu_{n_{j-1}} \leqslant m_\pi^{\alpha+1}\nu_{m_h}.$$

The definition of $\{m_k^{\alpha}\nu_{m_k}\}_{k\geqslant 1}$ implies

$$m_h^{\alpha}\nu_{m_h} \leqslant n_j^{\alpha}\nu_{n_j}.$$

and thus

$$m_h^{\alpha}\nu_{m_h} \leqslant n_j^{\alpha}\nu_{n_j} .$$

This yields

$$n_{j-1}^{\alpha+1}\nu_{n_{j-1}} \leqslant n_j^{\alpha+1}\nu_{n_j} .$$

The last inequality is contradicted by the choice of $n_j{}^{\alpha}\nu_{n_j}$.

PROPOSITION 4. *Let* $U : \mathscr{P}^l \to \mathscr{P}^l$ *be a continuous stationary linear operator. The condition* (12) *is equivalent to the following one: If for* $f \in \mathscr{P}^l$ *the equation* ${}^tUy = f$ *has solutions in* $\mathscr{P}'^l$ *then it also possesses a regular solution,* i.e. *a solution* $y \in \mathscr{P}^l$.

Proof. As we have seen, the restriction of ${}^tU$ to $\mathscr{V}^n$ is ${}^tC_{-n}$. It is immediate that if ${}^tC_{-n} \neq 0$, then the smallest nonzero eigenvalue of ${}^tC_{-n}^{*t}C_{-n}$ is λ_{-n}^2. The subspace $\mathscr{V}^n$ can be represented as an orthogonal sum

$$\begin{aligned} \mathscr{V}^n &= \mathscr{R}({}^tC_{-n}^*) \oplus \mathscr{N}({}^tC_{-n}), \\ \mathscr{V}^n &= \mathscr{R}({}^tC_{-n}) \oplus \mathscr{N}({}^tC_{-n}^*). \end{aligned} \tag{17}$$

Let D_n be the restriction of ${}^tC_{-n}$ to $\mathscr{R}({}^tC_{-n}^*)$. By Lemma 1, D_n is an isomorphism of $\mathscr{R}({}^tC_{-n}^*)$ onto $\mathscr{R}({}^tC_{-n})$ and $| D_n^{-1} | = 1/\lambda_{-n}$, if ${}^tC_{-n} \neq 0$.

Let $\mathscr{R}_3$, $\mathscr{N}_3$, $\mathscr{R}_4$ resp. $\mathscr{N}_4$, be closed in $\mathscr{P}'^l$ linear envelopes of $\mathscr{R}({}^tC_{-n}^*)$, $\mathscr{N}({}^tC_{-n})$, $\mathscr{R}({}^tC_{-n})$ resp. $\mathscr{N}({}^tC_{-n}^*)$. Obviously $\mathscr{N}_3$ is the kernel of ${}^tU$. Moreover,

$${}^tU\mathscr{P}'^l \subset \mathscr{R}_4 . \tag{18}$$

Assume that (12) holds and let φ be an element belonging to $\mathscr{P}^l$, $\varphi = \sum_n c_n e^{int}$, so that the equation ${}^tUy = \varphi$ has solutions in $\mathscr{P}'^l$. By (18) $\varphi \in \mathscr{R}_4$ and thus $c_n \in \mathscr{R}({}^tC_{-n})$. If $b_n = D_n^{-1}c_n$, then $b_n \in \mathscr{R}({}^tC_{-n}^*)$. We put $\psi = \sum_n b_n e^{int}$. If $b_n \neq 0$, then

$$| b_n | = | D_n^{-1}c_n | \leqslant | D_n^{-1} || c_n | \leqslant \frac{1}{\lambda_{-n}} | c_n | \leqslant K | n |^p | c_n | ,$$

and thus $\psi \in \mathscr{P}^l$. Furthermore, we have

$${}^tU\psi = \sum_n {}^tC_{-n}b_n e^{int} = \sum_n D_n b_n e^{int} = \sum_n c_n e^{int} = \varphi.$$

This means that the equation ${}^tUy = \varphi$ also has solutions belonging to $\mathscr{P}^l$.

Reciprocally, assume that if for $\varphi \in \mathscr{P}^l$ the equation ${}^tUy = \varphi$ has solutions in $\mathscr{P}'^l$, then it also has a solution $y \in \mathscr{P}^l$. We show that (12) holds. For $q \geqslant 0$ we put

$$\mu_q = \inf\{| n |^q \lambda_{-n} , \lambda_{-n} \neq 0, | n | \geqslant 1\}.$$

There is p, so that $\mu_p \neq 0$. Suppose not:

$$\mu_q = 0, \qquad \text{for} \qquad q = 0, 1, 2, \ldots .$$

We denote

$$\mu_q^{+} = \inf\{n^q \lambda_{-n}, \lambda_{-n} \neq 0, n \geqslant 1\}$$

$$\mu_q^{-} = \inf\{| n |^q \lambda_{-n}, \lambda_{-n} \neq 0, n \leqslant -1\}.$$

Obviously $\mu_q = \min(\mu_q^{+}, \mu_q^{-})$. There is a strictly increasing sequence of integers $q_h \geqslant 0$ so that $\mu_{q_h}^{+} = 0$ $(h = 1, 2, \ldots)$, or $\mu_{q_h}^{-} = 0$ $(h = 1, 2, \ldots)$.

Assume, for instance, $\mu_{q_h}^{+} = 0$ $(h = 1, 2, \ldots)$. Then $\mu_q^{+} = 0$ $(q = 1, 2, \ldots)$. From every sequence $\{n^q \lambda_{-n}\}_{n \geqslant 1}$ we choose the canonical subsequence $\{n_{qk}^q \lambda_{-n_{qk}}\}_{k \geqslant 1}$. By Lemma 2 the sequence of indices $\{n_{q+1,k}\}_{k \geqslant 1}$ is a subsequence of $\{n_{qk}\}_{k \geqslant 1}$ and thus $\{n_{hk}\}_{k \geqslant 1} \subset \{n_{qk}\}_{k \geqslant 1}$ for any $h \geqslant q$. We consider now the diagonal sequence $\{n_{hh}\}_{h \geqslant 1}$. If $h \geqslant q$, then $n_{hh} \in \{n_{qk}\}_{k \geqslant 1}$. For $\{n_{qk}^q \lambda_{-n_{qk}}\}_{k \geqslant 1}$ are canonical subsequences we have

$$\lim_{k \to \infty} n_{qk}^q \lambda_{-n_{qk}} = 0.$$

It follows that

$$\lim_{h \to \infty} n_{hh}^q \lambda_{-n_{hh}} = 0, \qquad \text{for any integer} \quad q \geqslant 1. \tag{19}$$

We define the two-sided sequence of vectors $\{d_n\}$ as follows: $d_{-n_{hh}} \neq 0$, $d_{-n_{hh}} \in \mathscr{R}({}^tC_{-n_{hh}})$ with

$$| d_{-n_{hh}} | = 1, \qquad {}^tC^*_{-n_{hh}} {}^tC_{-n_{hh}} d_{-n_{hh}} = \lambda^2_{-n_{hh}} d_{-n_{hh}}$$

and $d_n = 0$, for $n \neq -n_{hh}$. This choice is always possible. We put

$$g = \sum_n d_n e^{int}.$$

Obviously $g \in \mathscr{P}'^l$ and $g \notin \mathscr{P}^l$. We put

$$f = {}^tUg = \sum_n {}^tC_{-n} d_n e^{int}.$$

Obviously

$${}^tC_{-n}\, d_n = 0, \qquad \text{for} \qquad n \neq -n_{hh}$$

and

$$| {}^tC_{-n_{hh}} d_{-n_{hh}} |^2 = ({}^tC_{-n_{hh}} d_{-n_{hh}}, {}^tC_{-n_{hh}} d_{-n_{hh}})$$

$$= ({}^tC^*_{-n_{hh}} {}^tC_{-n_{hh}} d_{-n_{hh}}, d_{-n_{hh}}) = \lambda^2_{-n_{hh}} | d_{-n_{hh}} |^2 = \lambda^2_{-n_{hh}}.$$

From (19) we deduce $f \in \mathscr{P}^l$. The equation ${}^tUy = f$ has the solution $y = g$ in $\mathscr{P}'^l$. Since the kernel of U is $\mathscr{N}_3$, any other solution of ${}^tUy = f$ has the following form:

$$g_1 = g + \sum_n d'_n e^{int} = \sum_n (d_n + d'_n)\, e^{int},$$

where $d'_n \in \mathscr{N}({}^tC_{-n})$. By (17) we have

$$| d_{-n_{hh}} + d'_{-n_{hh}} |^2 = | d_{-n_{hh}} |^2 + | d'_{-n_{hh}} |^2 = 1 + | d'_{-n_{hh}} |^2 \geqslant 1$$

and thus $g_1 \notin \mathscr{P}^l$. Thus $f \in \mathscr{P}^l$, the equation ${}^tUy = f$ has solutions in $\mathscr{P}'^l$, but has no solution belonging to $\mathscr{P}^l$, which is a contradiction!

This implies there is an integer $p \geqslant 0$, so that $\mu_p \neq 0$. Then

$$| n^p | \lambda_{-n} \geqslant \mu_p, \qquad \text{for} \qquad n, \lambda_{-n} \neq 0$$

and condition (12) holds.

Remarks (a) Assume that (12) holds by [*1*] a 2π-periodic function $f(t)$ is analytic if and only if there are $\alpha > 0, \beta > 0$ so that the Fourier coefficients verify

$$| c_n | \leqslant \alpha e^{-\beta |n|}, \qquad \text{for any integer } n.$$

By the means of Proposition 4 one can show that if $f(t)$ is analytic and ${}^tUy = f$ has solutions belonging to $\mathscr{P}'^l$, then this equation also has an analytic solution.

(b) The following conditions are equivalent:

(i) If ${}^tUy \in \mathscr{P}^l$, then $y \in \mathscr{P}^l$;

(ii) (12) holds and the kernel of tU is of finite dimension.

From Banach's theorem and Propositions 3 and 4 we obtain

THEOREM 2. *Let $U : \mathscr{P}^l \to \mathscr{P}^l$ be a stationary continuous linear operator. The following conditions are equivalent*:

(i) *The equation ${}^tUy = f$ has solutions if and only if f is orthogonal to the solutions (in $\mathscr{P}^l$) of the equation $Ux = 0$;*

(ii) *There are $K > 0, p \geqslant 0$ so that*

$$\lambda_n \geqslant K | n |^{-p}, \qquad \textit{for any} \quad n, \lambda_n \neq 0;$$

(iii) *If for $f \in \mathscr{P}^l$ the equation ${}^tUy = f$ has solutions in $\mathscr{P}'^l$, then it also has a regular solution*, i.e. *a solution $y \in \mathscr{P}^l$.*

4. Stationary Systems of Neutral Type

Let A, B, C be $l \times l$-matrices and $f \in \mathscr{D}'^l$. We look for the solutions, belonging to $\mathscr{D}'^l$, of the system of the neutral type

$$Dy + Ay + B\tau_h y + C\tau_h Dy = f. \tag{20}$$

At the same time we look for the solutions, belonging to $\mathscr{D}^l$, of the homogeneous system

$$-Dx + {}^tAx + {}^tB\tau_{-h}x - {}^tC\tau_{-h} Dx = 0. \tag{21}$$

Let $U : \mathscr{D}^l \to \mathscr{D}^l$ be the operator

$$U\varphi = -D\varphi + {}^tA\varphi + {}^tB\tau_{-h}\varphi - {}^tC\tau_{-h} D\varphi, \qquad \text{for} \qquad \varphi \in \mathscr{D}^l.$$

The system (21) can be written $Ux = 0$ and (20) can be written ${}^tUy = f$. U is the convolution by the operator $\check{R}$ where

$$\check{R} = -I\, D\delta + {}^tA\delta + {}^tB\delta_{-h} - {}^tC\, D\delta_{-h}\,,$$

The Fourier coefficients of $\check{R}$ are

$$C_n = -ine^{inh}(Ie^{-inh} + {}^tC) + {}^tA + {}^tBe^{inh}.$$

Notice that the C_n *are the values on the set of points* (in, n, *an integer*) *of the characteristic function of* (20). As a matter of fact this is also true in the general case. Let $\lambda_n{}^2$ be the smallest nonzero eigenvalue of $C_n^* C_n$ (if $C_n = 0$, we put $\lambda_n = 0$). By Theorem 2, we have

THEOREM 3. *The following assertions are equivalent*:

(i) *The system* (20) *has solutions belonging to* $\mathscr{D}'^l$, *if and only if* f *is orthogonal to the solutions belonging to* $\mathscr{D}^l$ *of* (21);

(ii) *There are* $K > 0, p \geqslant 0$ *so that*

$$\lambda_n \geqslant K \mid n \mid^{-p} \qquad \textit{for any} \quad n, \lambda_n \neq 0;$$

(iii) *If for* $f \in \mathscr{D}^l$ *the system* (20) *has solutions belonging to* $\mathscr{D}'^l$, *then it has also a regular solution*, i.e. *a solution* $y \in \mathscr{D}^l$.

The eigenvalues $\lambda_n{}^2$ are very complicated functions of the coefficients A, B, C and exhibit very peculiar behavior. We shall see by studying a single neutral equation ($l = 1$) that there are cases when condition (ii) does not hold. However, we give a sufficient condition, which always holds for differential systems ($B = C = 0$) and time lag systems ($C = 0$). Let Γ be the circle $|z| = 1$ of the complex plane.

THEOREM 4. *If the matrix* C *has no eigenvalues on* Γ, *then condition* (ii) *of Theorem* 3 *holds.*

Proof. Let $\mathscr{B}$ be the algebra of operators (matrices) $\mathscr{C}^l \to \mathscr{C}^l$ and

$$\mathscr{Q}(z) = Iz + {}^tC.$$

By hypothesis, for $\mathscr{Q}(z)$, $z \in \Gamma$ there is an $\epsilon_z > 0$ so that the elements of the open ball $S(z, \epsilon_z)$ of $\mathscr{B}$ with the center in $\mathscr{Q}(z)$ and the radius ϵ_z are invertible. We assume that ϵ_z is the greatest positive number having this property. Since Γ is compact, we have $\inf_{z\in\Gamma} \epsilon_z \neq 0$. Let ϵ_0 be a positive number so that $\epsilon_0 \leqslant \inf_{z\in\Gamma} \epsilon_z$ and $\mathscr{M}$ be the closed ϵ_0-neighborhood of the set

$$\{\mathscr{Q}(z), z \in \Gamma\}.$$

All the elements of $\mathscr{M}$ are obviously invertible. Moreover, $\mathscr{M}$ is closed and bounded in $\mathscr{B}$. Since $\mathscr{B}$ is a space of finite dimension, $\mathscr{M}$ is compact. Since the mapping $M \to | M^{-1} |$ of $\mathscr{M}$ into the set of positive numbers is continuous, there is $K_1 > 0$ so that

$$| M^{-1} | \leqslant K_1 , \qquad \text{for any} \qquad M \in \mathscr{M}. \tag{22}$$

We deduce that there is an integer $n_0 > 0$ so that

$$-(in)^{-1}\, e^{-inh} C_n \in \mathscr{M}, \qquad \text{for} \qquad | n | \geqslant n_0 .$$

For $| n | \geqslant n_0$, this implies C_n is invertible and, by (22), we deduce $| n | \, | C_n^{-1} | \leqslant K_1$. Then

$$\frac{1}{| C_n^{-1} |} \geqslant K \, | n | , \qquad \text{for} \quad | n | \geqslant n_0 . \tag{**}$$

where $K = 1/K_1$. It remains to remark that $\lambda_n = 1/| C_n^{-1} |$, for $| n | \geqslant n_0$.

Remarks. (a) If the condition of Theorem 4 holds, then the matrices C_n are invertible for $| n | \geqslant n_0$. According to Section 3, the homogeneous system ${}^tUy = 0$ associated to (20) has a finite set of linear independent solutions. The relation (**) makes us think that for f a continuous function, the solutions y of (20) possess a continuous derivative.

(b) If the time lag h is rational, then $\{e^{inh}\}$ is a finite set. In this case the condition of Theorem 4 can be replaced by the requirement that the matrix C should have no eigenvalue of the form e^{inh}.

We shall now discuss in detail some of the situations which appear in the case of a single equation of neutral type ($l = 1$). Further we assume that A, B, C are real numbers and thus ${}^tA = A$, ${}^tB = B$, ${}^tC = C$. The Fourier coefficients of $\check{R}$ are

$$C_n = -in(1 + Ce^{inh}) + A + Be^{inh} \tag{23}$$

and $\lambda_n = | C_n |$. At the same time, we suppose $0 < h < 2\pi$. This hypothesis does not restrict the generality, for if $h = 2k\pi + h_1$, $0 < h_1 < 2\pi$, (k = integer), then $\tau_h = \tau_{h_1}$ on $\mathscr{P}^l$ and $\mathscr{P}'^l$.

(1) $C \neq \pm 1$. In this case, by Theorem 4, condition (ii) of Theorem 3 holds.

(2) $C = -1$ and h is commensurable with 2π. In this case the sequence $\{e^{inh}\}$ is periodic. Let q be the smallest positive period. Then $C_n = A + B$, for $n = mq$. At the same time, there are $K > 0$ and an integer $n_0 > 0$ so that

$$| C_n | \geqslant K | n |, \qquad \text{for} \qquad | n | \geqslant n_0 , \quad n \neq mq.$$

Thus condition (ii) of Theorem 3 holds. Let us remark that if $A + B = 0$, then the kernels of U and ${}^tU$ are of infinite dimension and thus the homogeneous equation attached to (20) has generalized solutions.

(3) $C = 1$ and h is commensurable with 2π. This case can be discussed as case (2). We only remark that if the smallest positive period q of the sequence $\{e^{inh}\}$ is odd, then there are $K > 0$ and an integer $n_0 > 0$ so that

$$| C_n | \geqslant K | n |, \qquad \text{for all} \quad n, | n | \geqslant n_0 .$$

(4) $C = -1$, h incommensurable with 2π and $| A | \neq | B |$. In this case

$$\begin{aligned} C_n = A + B - 2 \left(B \sin \frac{nh}{2} + n \cos \frac{nh}{2}\right) \sin \frac{nh}{2} \\ + 2i \left(B \cos \frac{nh}{2} - n \sin \frac{nh}{2}\right) \end{aligned} \tag{24}$$

and

$$C_n = A - B + 2 \left(B \cos \frac{nh}{2} - n \sin \frac{nh}{2}\right)\left(\cos \frac{nh}{2} + i \sin \frac{nh}{2}\right). \tag{25}$$

Let n_0 be so that for $| n | \geqslant n_0$, we have

$$\frac{2}{| n |} \leqslant \frac{| A - B |}{2}, \quad \frac{2}{n^2} (| B | + | n |) \leqslant \frac{| A + B |}{2}, \quad \frac{2}{| n |^3} \leqslant \frac{| A + B |}{2}.$$

Suppose $| n | \geqslant n_0$. If $| \sin nh/2 | < n^{-2}$, then by (24) we deduce

$$\begin{aligned} | C_n | &\geqslant \left| A + B - 2 \left(B \sin \frac{nh}{2} + n \cos \frac{nh}{2}\right) \sin \frac{nh}{2} \right| \\ &\geqslant | A + B | - \frac{2}{n^2} (| B | + | n |) \geqslant \frac{| A + B |}{2} \geqslant \frac{2}{| n |^3}. \end{aligned}$$

If

$$\left| \sin \frac{nh}{2} \right| \geqslant \frac{1}{n^2} \qquad \text{and} \qquad \left| B \cos \frac{nh}{2} - n \sin \frac{nh}{2} \right| \geqslant \frac{1}{|n|},$$

then by (24) we obtain

$$|C_n| \geqslant 2 \left| B \cos \frac{nh}{2} - n \sin \frac{nh}{2} \right| \left| \sin \frac{nh}{2} \right| \geqslant \frac{2}{|n|^3}.$$

Finally, if

$$\left| \sin \frac{nh}{2} \right| \geqslant \frac{1}{n^2} \qquad \text{and} \qquad \left| B \cos \frac{nh}{2} - n \sin \frac{nh}{2} \right| < \frac{1}{|n|},$$

then by (25) we deduce

$$\begin{aligned} |C_n| &\geqslant |A - B| - 2 \left| B \cos \frac{nh}{2} - n \sin \frac{nh}{2} \right| \\ &\geqslant |A - B| - \frac{2}{|n|} \geqslant \frac{|A - B|}{2} \geqslant \frac{2}{|n|^3}. \end{aligned}$$

Eventually

$$|C_n| \geqslant \frac{2}{|n|^3}, \qquad \text{for every} \quad |n| \geqslant n_0$$

and condition (ii) of Theorem 3 holds.

(5) $C = 1$, h incommensurable with 2π and $|A| \neq |B|$. This case can be discussed as in case (4).

(6) $C = -1$, h incommensurable with 2π and $A = -B$. We have

$$|C_n| = 2 \, |in + B| \, |\sin n\pi\mu|,$$

where $\mu = h/2\pi$, thus $0 < \mu < 1$. For any integer n there is a single integer $m = m(n)$, so that

$$-\tfrac{1}{2} \leqslant n\mu - m < \tfrac{1}{2}. \tag{26}$$

Obviously $n \cdot m \geqslant 0$. We have

$$|C_n| = 2 \, |in + B| \, |\sin \pi(n\mu - m)|$$

and by (26),

$$\pi \, |in + B| \, |n\mu - m| \leqslant |C_n| \leqslant 2\pi \, |in + B| \, |n\mu - m|$$

or

$$\pi \, |in + B| \, |n| \left| \mu - \frac{m}{n} \right| \leqslant |C_n| \leqslant 2\pi \, |in + B| \, |n| \left| \mu - \frac{m}{n} \right|,$$

$$\text{for} \quad n \neq 0. \tag{27}$$

We are thus conducted to a problem concerning the approximation of the irrational number μ by rational numbers. We distinguish two subcases:

(6_1) μ, such that there are $K_1 > 0$ and $p \geqslant 0$ with

$$\left| \mu - \frac{q}{n} \right| \geqslant \frac{K_1}{|n|^p}, \text{ for any positive integers } q, n (n \neq 0). \tag{28}$$

By (27)

$$|C_n| \geqslant \pi \left| i + \frac{B}{n} \right| \frac{K_1}{|n|^{p-2}}, \qquad \text{for every} \quad n \neq 0$$

and thus condition (ii) of Theorem 3 holds.

(6_2) μ, such that there are no $K_1 > 0$ and $p \geqslant 0$ so that (28) holds. We remark that by [5], the set of these numbers $\mu \in [0, 1]$ is of zero Lebesgue's measure and thus the corresponding set of time lags $h = 2\pi\mu$ is also of zero Lebesgue's measure. However, as it is shown in the theory of approximation of irrational numbers, this kind of numbers μ exists. In case (6_2), for any $K_1 > 0$ and $p \geqslant 0$, the inequality

$$\left| \mu - \frac{q}{n} \right| < \frac{K_1}{|n|^{p+2}}$$

has an infinite set of solutions (q, n), q, $n > 0$. Obviously this set contains pairs for which n is as great as one likes. This yields that there is an infinite set of solutions (q, n) which also verify $|n\mu - q| < 2^{-1}$. Let (m_j, n_j) be these pairs. Obviously, they verify (26). By (27) we deduce

$$|C_{n_j}| < \frac{2\pi(1 + |B|) K_1}{|n_j|^p}.$$

Therefore for any $K > 0$ and $p \geqslant 0$ there are integers n so that

$$|C_n| < \frac{K}{|n|^p}$$

In case (6_2) *condition* (ii) *of Theorem* 3 *is not verified.*

(7) $C = 1$, h incommensurable with 2π and $A = B$. This case can be discussed as in case (6).

In the last two cases (8) $C = -1$, h incommesurable with 2π, $A = B$ and (9) $C = 1$, h incommensurable with 2π, $A = -B$ it is possible to show that condition (*ii*) of Theorem 3 holds for almost every $h \in [0, 2\pi)$.

Remarks. (a) It seems interesting to point out that in case (6) Eq. (20) can be related to a functional equation considered in [*1*] in connection with other problems. In fact, in case (6), Eq. (20) can be written as $Dz - Bz = f$, where $y - \tau_h y = z$. This functional equation was considered in other problems in [*1*].

(b) In case (6_1) for any integer $s \geqslant 1$, one can choose an irrational μ and a continuous perturbation f so that Eq. (20) has only solutions which are distributions of order $\geqslant s$.

(c) In case (6_2) the kernel of U and ${}^tU$ are zero and thus any $f \in \mathscr{P}'$ is orthogonal to the kernel of U. However, by Theorem 3 there is $f \in \mathscr{P}'$ so that Eq. (20) has no solutions. Moreover, if the irrational μ can be sufficiently well approximated by rational numbers (and there are irrational numbers which can be approximated as well as one likes, see [*5*]), then we can even choose f belonging to $\mathscr{P}$ so that Eq. (20) has no solutions belonging to $\mathscr{P}'$. This shows that the assertions of [*6*] are inexact.

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